J. Fluid Mech. (1983), *vol.* 134, *pp.* 91–101
Printed in Great Britain

Anomalous refraction and conjugate solutions of finite-amplitude water waves

By D. H. PEREGRINE AND S. C. RYRIE†
School of Mathematics, University of Bristol, Bristol BS8 1TW, England

(Received 14 April 1982 and in revised form 28 April 1983)

Calculations of the refraction of water waves obliquely incident on a beach indicate that in certain circumstances finite-amplitude waves are refracted to turn in a sense opposite to the refraction of linear waves. This is termed 'anomalous refraction'. It is demonstrated that similar solutions exist for a wide class of weakly nonlinear dispersive waves. When anomalous refraction solutions exist there are two 'conjugate' solutions satisfying the slowly varying wave equations. Properties of the conjugate solutions are given here. Discussion of the possibility of jumps in wave properties between conjugate solutions and their relevance to refraction is in another paper (Peregrine 1983), which shows that the anomalous-refraction solution is not normally relevant on a beach.

1. Introduction

Refraction calculations for periodic steady wavetrains obliquely incident onto a beach with parallel contours are presented in Ryrie & Peregrine (1982). The waves are assumed to behave locally like a uniform wavetrain; 'numerically exact' integral properties of finite-amplitude wavetrains are used in the calculations. The work extends that of Stiassnie & Peregrine (1980) for normally incident waves. For most of the examples illustrated in the above papers the finite-amplitude solutions depart little from the corresponding linear-wave solutions except in the neighbourhood of the highest waves.

There are two categories of solution where there is a marked departure from linear solutions. One is near the position of linear caustics where waves are 'trapped' on the shallower side of the caustic. The finite-amplitude solutions have a limiting singular point similar to that found for R-type caustics in Peregrine & Smith (1979) and Peregrine & Thomas (1979). A second branch of the solution corresponding to steeper waves meets the original solution at the singularity.

The other solutions which differ strongly from linear solutions are those for which the waves are incident nearly parallel to the bottom contours. These display 'anomalous refraction'. Instead of being refracted by shallower water to propagate more nearly normal to the bottom contours, as one would expect from linear solutions, these solutions show that the waves' propagation direction turns more nearly parallel to the bottom contours as depth decreases. Anomalous refraction corresponds to the second solution near a caustic. The range of incident angles in which these solutions exist depends on the wave steepness and is limited by the waves of maximum steepness.

† Present address: Department of Computer Studies and Mathematics, Bristol Polytechnic, Bristol BS16 1QY.

All the anomalous refraction solutions have 'conjugate' solutions displaying normal refraction for which all the conserved quantities defining a solution have the same values. At any given depth the pairs of conjugate wavetrains are well defined, and examples are given here. Extensions of this property to consideration of jumps of wave properties and to the nature of wave fields in the neighbourhood of 'linear caustics' are given in another paper (Peregrine 1983).

2. Anomalous refraction

Refraction solutions for finite-amplitude water waves described here are found in the same manner as those in Ryrie & Peregrine (1982) by using Whitham's method of averaging for nonlinear waves. The wave field is assumed to be steady and to depend only on the coordinate perpendicular to the depth contours, which are taken to be parallel. As indicated in Ryrie & Peregrine (1982), certain solutions for obliquely incident waves show behaviour which is qualitatively different from that for linear-wave solutions.

Figures 1 (a, b) illustrate the different types of behaviour. These show how waves of a given steepness in deep water refract, for different angles of incidence, as they propagate into shallower water. The solutions for angles of incidence less than a certain critical angle are very close to the corresponding linear theory. The waves turn towards the shore and initially show a reduction in amplitude as the incident wave-action flux spreads out. Solutions for greater angles of incidence show opposite properties: the waves are turned more nearly parallel to the shoreline and grow in steepness. In the case shown, with initial steepness $a_\infty k_\infty = 0.271$, the critical angle at which this change occurs is approximately $\theta_\infty = 77°$. Thus a wavetrain for which $a_\infty k_\infty = 0.271$ shows regular refraction for $\theta_\infty < 77°$ and anomalous refraction for $\theta_\infty > 77°$. Here a is the wave amplitude, k the wavenumber, θ the angle of incidence ($\theta = 0$ is normal incidence) and subscript ∞ denotes deep-water values.

This sudden switch in behaviour can be put in perspective by plotting the conjugate solutions, which have the same frequency, longshore wavenumber and onshore flux of wave action as the solutions illustrated in figure 1. There is a continuous family of solutions across the critical angle for all finite water depths. We bring the deep-water case to a finite position by plotting wave amplitude against $\tanh k_\infty h$. It is then possible to sketch connections between conjugate solutions across the non-physical part of the diagram where $\tanh k_\infty h > 1$, as illustrated in figure 2. The depth of the bed is denoted by h. For $\theta \leqslant 60°$ no branch for anomalous refraction is shown, since it would correspond to waves with more than the maximum wave-action flux.

If one considers waves commencing with a given steepness, in a finite depth of water, a diagram very similar to figure 2 can be prepared. In that case connections between conjugate solutions can be made with real solutions. However, the solution method clearly fails at the critical angle, and near any point where a solution has a vertical tangent. Except in the deep-water case, such singularities are in the neighbourhood of a linear caustic. Examples are shown in Ryrie & Peregrine (1982, figures 6 and 7); see also the corresponding cases for waves on a shearing current in §4 of Peregrine & Thomas (1979), especially figure 3, and waves approaching a circular caustic (Peregrine 1981).

The two solutions can be better understood by examining the solution method. Let Ox_1 be normal to the depth contours and Ox_2 be parallel to them. For a mean current $\boldsymbol{U}^*$, and steady incident waves of wavenumber $\boldsymbol{k}^*$ at an angle θ to Ox_1, the following

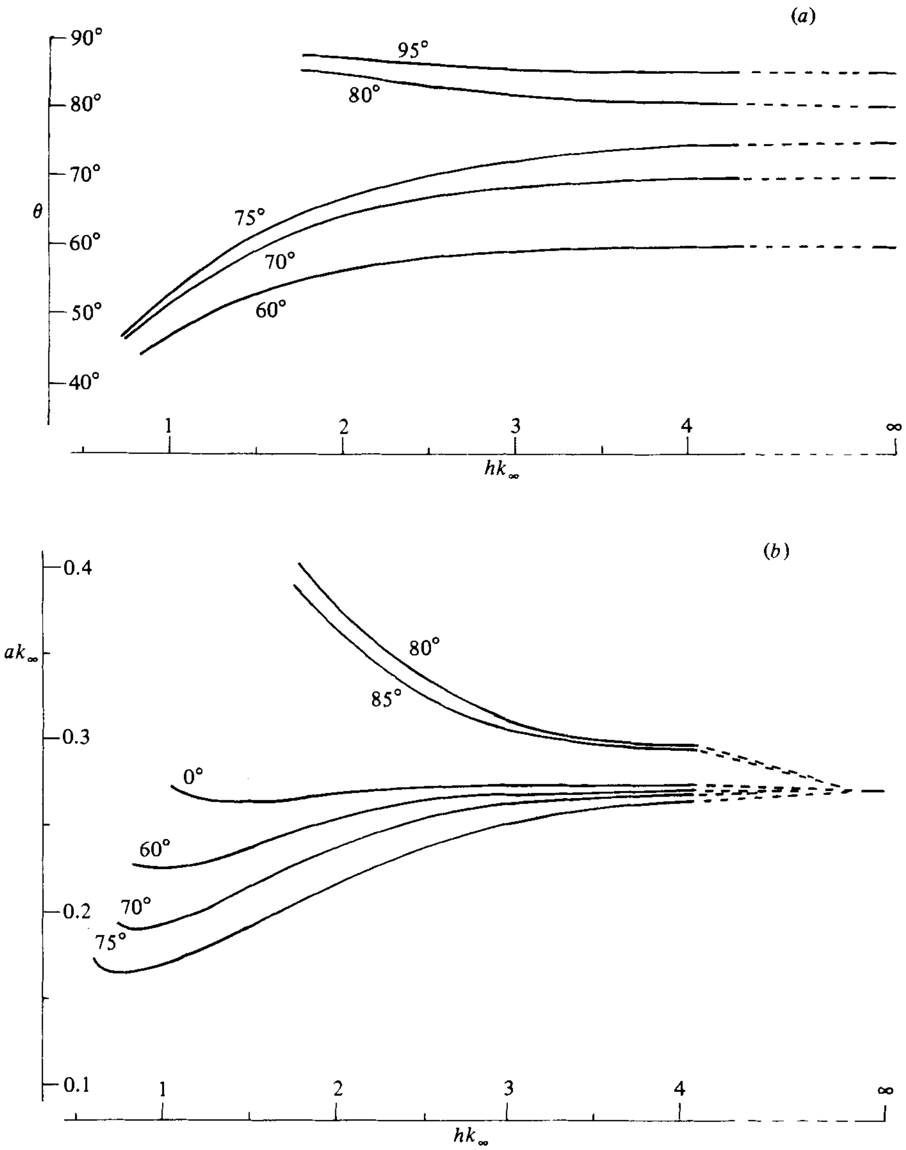

FIGURE 1. Variation of wave properties with depth h, to bed, for deep-water steepness $a_\infty k_\infty = 0.271$. Each solution curve is labelled with the value of θ at infinity. The solutions for 85° and 80° exhibit anomalous refraction. (*a*) Wave direction θ. (*b*) Wave amplitude ak_∞.

equations are obtained (Ryrie & Peregrine 1982, equations (10)–(15)): the longshore wavenumber component

$$k_2^* = k^* \sin\theta = \text{constant}; \tag{1}$$

the wave frequency

$$\omega^* = c^*k^* + \boldsymbol{U}^*\cdot\boldsymbol{k}^* = \text{constant}; \tag{2}$$

the onshore mass flow

$$Q_1^* = \rho^* D^* U_1^* + I_1^* = \text{constant}; \tag{3}$$

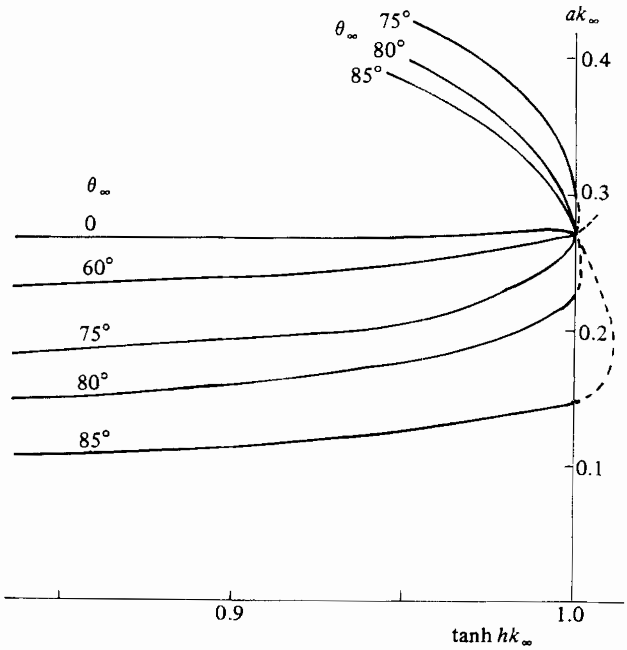

FIGURE 2. Variation of wave amplitude with depth for deep-water steepness $a_\infty k_\infty = 0.271$ together with corresponding conjugate solutions (broken lines connect conjugate solutions, $\tanh hk_\infty$ is used as abscissa to bring $h = \infty$ to a finite point). Although the broken lines do not represent solutions, if initial conditions were in finite depth of water a similar pattern of solution curves would result.

the onshore wave-action flux

$$B_1^* = \frac{U_1^* I^*}{k^*} + (3T^* - 2V^* + \tfrac{1}{2}\rho D\overline{u_b^{*2}}) \frac{k_1^*}{k^{*2}} = \text{constant} \tag{4}$$

(correcting a typographical error in Ryrie & Peregrine); and

$$\gamma^* = g^*(D^* - h^*) + \tfrac{1}{2}U^{*2} + \tfrac{1}{2}\overline{u_b^{*2}} = \text{constant}; \tag{5}$$

where c^* is the waves' phase velocity, given by the dispersion equation, ρ^* is water density, g^* is the acceleration due to gravity, D^* is the water depth, I^* is the mean mass flow associated with the waves, T^*, V^* are the mean kinetic and potential energies per unit area of the wave motion, $\overline{u_b^{*2}}$ is the mean-square wave-induced velocity at the bed, and h^* is the depth of the bed below a fixed reference level, such as mean water level in deep water.

Equation (5) is a mean Bernoulli equation which comes directly from Whitham's averaged Lagrangian (Whitham 1974, §6), or from the mean-flow momentum equations of Stiassnie & Peregrine (1979). In either case the flow must be irrotational.

For zero longshore current, $U_2 = 0$, (1)–(4) can be condensed to

$$m_2[c - \frac{I}{D}\cos^2\theta]^2 = \sin\theta, \tag{6}$$

$$\left(-\frac{I^2}{D} + 3T - 2V + \tfrac{1}{2}D\overline{u_b^2}\right)\cos\theta\sin^3\theta = b_1 m_2^3, \tag{7}$$

where the absence of a $*$ from previously defined variables implies that they are made dimensionless with ρ^*, g^* and k^*, and

$$b_1 = \frac{B_1^* \omega^{*6}}{\rho^* g^*}, \quad m_2 = \frac{g^* k_2^*}{\omega^{*2}}, \tag{8), (9}$$

and hence

$$\frac{gk^*}{\omega^{*2}} = m_2 \operatorname{cosec} \theta. \tag{10}$$

For linear waves, c is independent of wave steepness and is a known function of the water depth which is not changed in that approximation. I/cD is also negligible, so that (6) is used to find θ and (7) gives the wave steepness.

For nonlinear waves, c depends on wave steepness and I/cD is no longer negligible (Cokelet's (1977) tables indicate it does not exceed 0.05). Equations (6) and (7) are simultaneous equations for wave steepness and direction – details of solution methods are in Ryrie & Peregrine (1982). The existence of two distinct solutions of (6) and (7) is shown most readily for waves in deep water. There $I/D = 0, \frac{1}{2}D\overline{u_b^2} = 0$, so that θ may be eliminated to give a single equation for wave steepness:

$$c^6(3T-2V)(1-m_2^2 c^4)^{\frac{1}{2}} = b_1. \tag{11}$$

The left-hand side of this equation, for given m_2, is a function of wave steepness with the following properties. It is zero for zero wave steepness since $3T-2V$ is a dimensionless form of the magnitude of the wave-action flux. It is also zero when $c^2 = m_2^{-1}$. Since $3T-2V$ and c are both smoothly increasing functions of wave steepness for the realistic range of their values, the left-hand side of (11) is a smooth and positive function between its two zero values. Thus for some steepness it has a maximum value. Hence two solutions of (11) exist for values of b_1 between zero and some maximum value b_{max} at which the roots coincide. For $b_1 > b_{max}$ there are no solutions. A similar result holds for finite water depths but is less easy to demonstrate. We refer to such pairs of solutions as 'conjugate' solutions.

Two water-wave properties modify the above argument. For given frequency and water depth, waves have a maximum phase velocity and maximum steepness. Thus the solution of (11) corresponding to a steeper wave may not always exist. Also there may be further solutions possible with steepness in the interval between that of maximum phase velocity and maximum steepness. We do not discuss these solutions since (*a*) wavetrains in that steepness range on deep water are unstable to a rapidly growing perturbation (Longuet-Higgins 1978) and it seems likely that the same is true for finite water depths; and (*b*) unlike other aspects of this problem, such behaviour is probably peculiar to surface water waves.

The existence of anomalous refraction is seen to be due to the variation of wave phase velocity with steepness. The differing character of ordinary and anomalous refraction may roughly be described as follows.

Ordinary refraction: decreasing depth leads to a lower phase velocity which implies that wave crests turn to become more nearly parallel to the depth contours.

Anomalous refraction: decreasing depth leads to shorter *steeper* waves such that the phase velocity increases, implying that wave crests turn to become more nearly perpendicular to the depth contours.

At the critical angle these two effects balance each other.

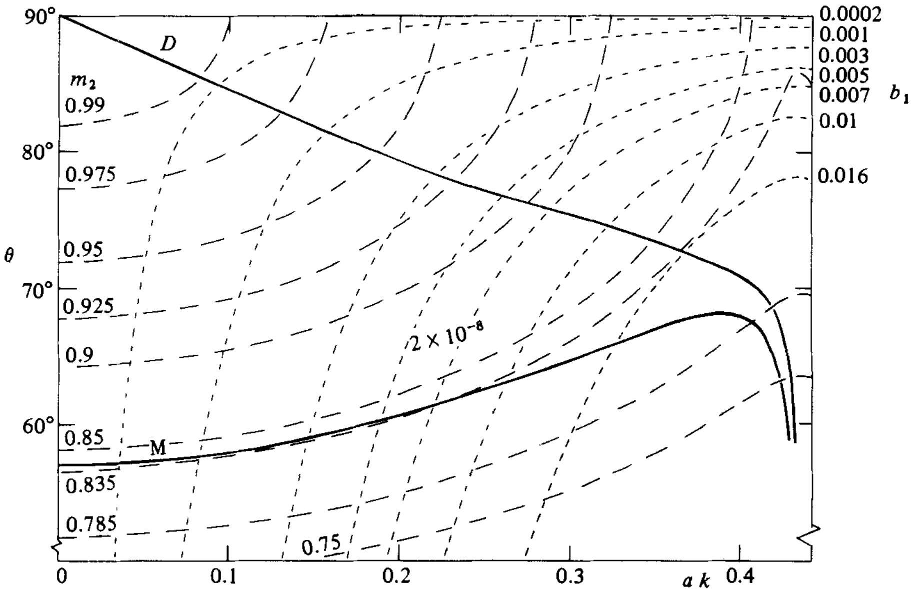

FIGURE 3. Conjugate solutions for deep-water waves may be identified from this diagram; they correspond to the two points of intersection of a line of constant m_2 (broken lines) and a line of constant b_1 (dashed lines). Line D is the locus of self-conjugate solutions and line M is the locus of solutions conjugate to the highest wave. The line $b_1 = 0$ consists of $ak = 0$ and $\theta = 90°$.

3. Conjugate solutions

The conjugate solutions found in the refraction calculations described above exist for waves in a uniform medium independent of any non-uniformities in the medium or the waves. In a uniform medium the definition of x_1 and x_2 directions become arbitrary; however, if given components of wave-action flux and wavenumber are to be conserved, for a given frequency, then these components define two directions Ox_1, Ox_2 with components B_1^* and k_2^* given. Although we take these directions to be orthogonal, the effects we describe do not depend on this.

Having chosen the coordinate directions, and medium properties, the whole range of plane-wave solutions of a given frequency can be described by two parameters, in our case wave steepness ak and direction θ. Here, to be consistent with the refraction calculations θ is the angle between Ox_1 and $\boldsymbol{k}^*$. Thus Ox_2 corresponds to the direction of depth contours, or to the direction of a linear caustic, $\theta = \frac{1}{2}\pi$. We are thus particularly interested in values of θ near $\frac{1}{2}\pi$. Figure 3 shows part of the (ak, θ)-plane for deep water waves.

Two families of lines are drawn in figure 3. One set is for values of the dimensionless wave-action flux component b_1; note that the lines $b_1 = 0$ are both $ak = 0$ and $\theta = \frac{1}{2}\pi$. The second set is for values of m_2. It is the increase of wavelength with steepness for fixed frequency ω^* which causes these lines to curve as ak increases. Linear theory gives the straight line at $ak = 0$.

If one wave state is selected by choosing a point in the (ak, θ)-plane, its conjugate wave state is readily found by following the b_1 and m_2 lines through that point to their second point of intersection. (Further points of intersection can occur for the steepest waves, but as stated earlier these are not considered here.)

Consideration of a few examples soon leads to the identification of two important

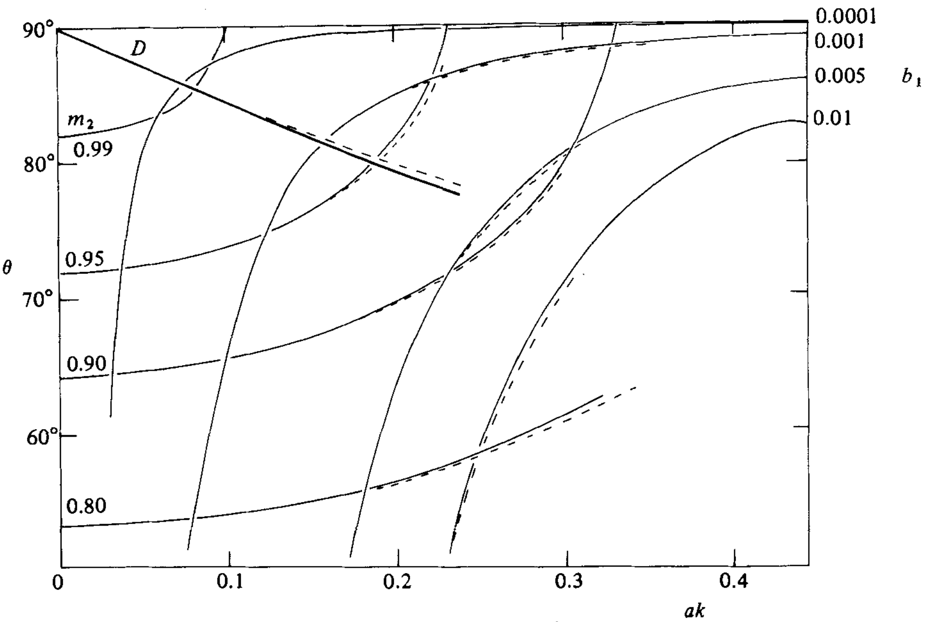

FIGURE 4. This figure is a simplified version of figure 3 with results from a near-linear approximation added for comparison. Dashed lines are the near-linear curves.

boundaries, which are marked in the diagram. One, which we call line M, corresponds to waves whose conjugate solution is a wavetrain of maximum phase velocity. Waves in the (ak, θ)-plane lying below this line have no conjugate solution.

The other boundary in the diagram, line D, goes through points that are 'self-conjugate'. That is they correspond to the double roots of (11). Line D divides the (ak, θ)-plane into two regions corresponding to the two cases of regular and anomalous refraction. On the line D, the Jacobian $\partial(b_1, m_2)/\partial(ak, \theta)$ vanishes, and this provides a way of determining the line.

A near-linear approximation to figure 3 can readily be made by using the approach followed by Peregrine & Smith (1979). In their notation, the dispersion equation is

$$G(\omega, \boldsymbol{k}) + H(\boldsymbol{k})\, a^2 = 0, \tag{12}$$

and the wave-action flux is

$$\boldsymbol{B} = G_{\boldsymbol{k}} a^2 + \tfrac{1}{2} H_{\boldsymbol{k}} a^4. \tag{13}$$

The functions $G(\omega, \boldsymbol{k})$ and $H(\boldsymbol{k})$ need have no special form. Conjugate solutions occur near an R-type caustic, which Peregrine & Smith (1979) show can occur for wave systems for which

$$H/G_{k_1 k_1} > 0. \tag{14}$$

This condition can be satisfied in a wide range of wave systems, which hence have conjugate solutions.

The near-linear expressions (12) and (13) can easily be used for deep-water waves in (2) and (4). A comparison between near-linear and 'numerically exact' results is given in figure 4. This shows the (ak, θ)-plane again with lines drawn using near-linear theory and corresponding lines from figure 3. It appears that the near-linear theory is adequate up to a steepness of $ak = 0.2$.

The results presented in figure 3 have parallels for water waves on any given finite

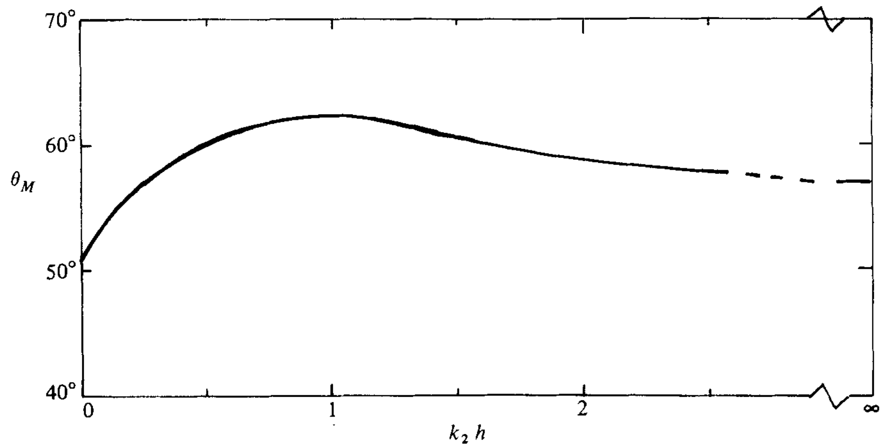

FIGURE 5. The angle at which waves of zero amplitude are conjugate to waves of maximum steepness as a function of depth.

depth of water. The major quantitative change is that the range of θ for which conjugate solutions exist changes as the depth increases. To illustrate this, figure 5 shows the angle θ_M at which the line M meets $ak = 0$, as a function of dimensionless depth $k_0 h$, where k_0 is the wavenumber of linear waves. For this point, $b_1 = 0$, so that it is easily deduced from (7) that $\theta = \frac{1}{2}\pi$ for the highest wave, and from (1) and (2) that

$$\sin\theta_M = \frac{c_0^*}{c_{\max}^*}, \tag{15}$$

where c_0^* and $c_{\max}^*$ are the phase velocity for infinitesimal waves and the maximum phase velocity respectively. Depths are related by (5).

An indication of how line D changes for different depths is obtained by considering its behaviour in the limit as $ak \to 0$, $\theta \to \frac{1}{2}\pi$. After some analysis, (6) and (7) yield

$$\tfrac{1}{2}\pi - \theta = \left(\frac{2c_0'}{c_0}\right)^{\frac{1}{2}} ak + O[a^2k^2, (\tfrac{1}{2}\pi - \theta)^2], \tag{16}$$

where c_0 is the linear phase velocity and

$$c_0' = \frac{\mathrm{d}c}{\mathrm{d}(a^2k^2)}$$

evaluated at $ak = 0$. For Stokes waves

$$\frac{2c_0'}{c_0} = \frac{9 - 10\tau^2 + 9\tau^4}{8\tau^4}, \quad \text{where} \quad \tau = \tanh k_0 h. \tag{17}$$

Figure 6 shows the variation of $(2c_0'/c_0)^{\frac{1}{2}}$ with $k_0 h$. The large growth of this coefficient as $k_0 h \to 0$, parallels the failure of the Stokes approximation in that limit. The refraction calculations of Stiassnie & Peregrine (1980) and Ryrie & Peregrine (1982) make use of a 'train of solitary waves' approximation for small values of $k_0 h$, but this approximation fails as $ak \to 0$.

For small-amplitude waves, Stokes waves can be matched with cnoidal waves when the Ursell number a/k^2D^3 is $O(1)$; for example see Flick, Guza & Inman (1981), where the matching of phase velocity and energy flux is discussed. For sufficiently long

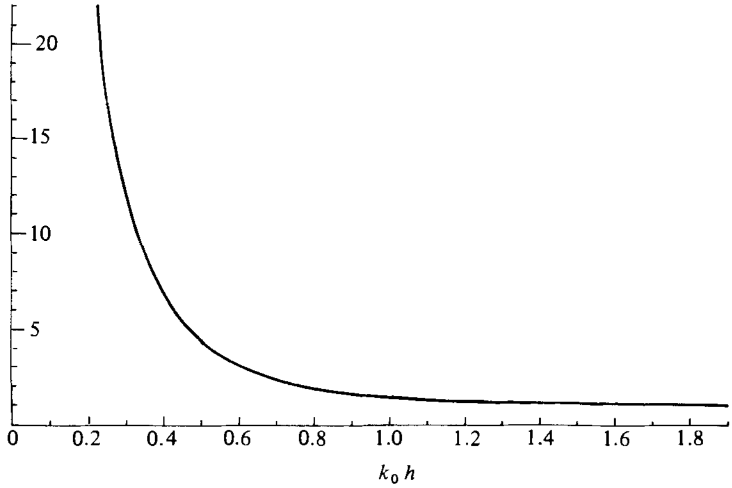

FIGURE 6. Slope of line D at $(0, \frac{1}{2}\pi)$ in the (ak, θ)-plane as a function of water depth.

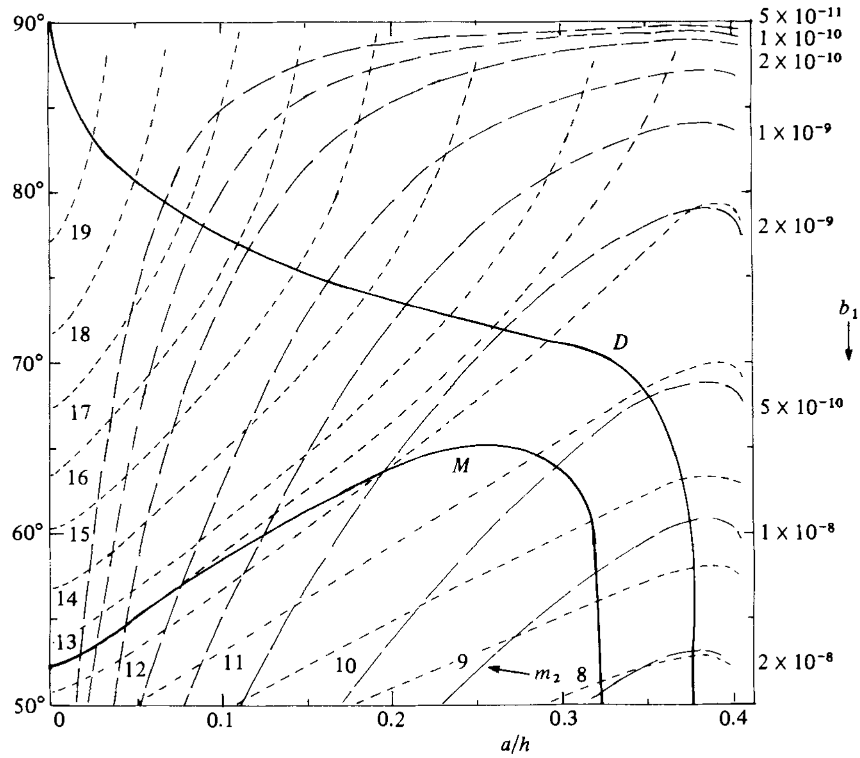

FIGURE 7. Conjugate solutions may be found from this diagram of the $(a/h, \theta)$-plane for waves with $k_2^* h^* = 0.05$. See caption to figure 3.

waves the velocity increases directly with the amplitude, and line D will be tangential to $ak = 0$ in that case, with

$$(\tfrac{1}{2}\pi - \theta)^2 \propto ak.$$

In this case, as the waves approach a solitary-wave type of solution, the parameter a/h is a better measure of wave steepness. An example of the $(a/h, \theta)$-plane for $k_2^* h^* = 0.05$ is given in figure 7.

The fact that line D meets $ak = 0$ in figures 3 and 7 implies that linear theory may be extremely limited in application to refraction problems in cases where waves travel parallel to depth contours. The domain of the steeper, essentially nonlinear, solution extends to $ak = 0$ at $\theta = \frac{1}{2}\pi$. Peregrine (1983) shows that it is necessary to consider effects due to nonlinear splitting of the group velocity in these cases, and that diffraction is also likely to be significant.

5. Discussion

The existence of conjugate solutions has not previously been noted. They raise the possibility of jumps in wave properties between two conjugate wave states. The possibility of wave jumps, or shocks, is noted in Whitham (1965) and other papers. Peregrine (1983) describes wave jumps between these solutions and shows that in a hydraulic analogy the anomalous solution corresponds to subcritical flow. Hence, it only occurs when some 'control' (like a weir in channel flow) affecting the wave field exists. For waves incident on a beach such a control does not usually occur, and an offshore influence on the wave field ensures that a normal refraction solution occurs. This is analogous to upstream influence in hydraulics. Offshore influence commences where waves first encounter the beach. This could either be at some initial time, or some alongshore point. In either case it is necessary to introduce the extra dimension, time or longshore distance, into the analysis, and the approach of this and the preceding papers is inapplicable. The transition from an 'anomalous' to a 'regular' solution appears to involve solutions with this extra dimension of time or space. Only the reverse, regular to anomalous, transition can be described by a jump. The case of weak variation in the Ox_2 direction is discussed in Peregrine (1983).

Line D of figure 3 appears to correspond to (i) line E of figure 2 in Saffman & Yuen (1980); (ii) line W of figure 9 in Crawford *et al.* (1981). These are neutral stability curves for modulations at an angle θ to uniform wavetrains of the corresponding steepness. Line D of figure 3 is not quite the same line as (i) and (ii) above since they are for waves of a given wavenumber, whereas line D is for given frequency. A version of figure 3 (not illustrated here) using a dimensionless wave-action flux

$$\beta_1 = \frac{B_1^* k^{*3}}{\rho^* g^*} = b_1 \left(\frac{g^* k^*}{\omega^{*2}}\right)^3 \tag{18}$$

instead of b_1 gives the lines (i) and (ii). However, g^*k^*/ω^{*2} is a function of wave steepness.

Figure 9 of Crawford *et al.* (1981) is of interest since it includes curves calculated by using a nonlinear Schrödinger equation and the Zakharov equation. The Zakharov equation allows for the effect of 'free' Fourier components as well as 'bound' components, and the nonlinear Schrödinger (NLS) equation is restricted to a small range of values of θ: both equations allow a greater rate of wave modulation than the present work but are not applicable to the steepest waves. Peregrine (1983) uses an NLS equation.

S. C. Ryrie acknowledges the support of the Science and Engineering Research Council and D. H. Peregrine thanks the La Jolla Foundation for Earth Sciences for the opportunity to visit the Institute of Geophysics and Planetary Physics at the University of California, San Diego, where this paper was prepared. Referees are thanked for helpful comments.

REFERENCES

COKELET, E. D. 1977 Steep gravity waves in water of arbitrary uniform depth. *Phil. Trans. R. Soc. Lond.* A **286**, 183–230.

CRAWFORD, D. R., LAKE, B. M., SAFFMAN, P. G. & YUEN, H. C. 1981 Stability of weakly nonlinear deep-water waves in two and three dimensions. *J. Fluid Mech.* **105**, 177–191.

FLICK, R. E., GUZA, R. T. & INMAN, D. L. 1981 Elevation and velocity measurements of laboratory shoaling waves. *J. Geophys. Res.* **86**, C5, 4149–4160.

LONGUET-HIGGINS, M. S. 1978 The instabilities of gravity waves of finite amplitude in deep water. I. Super-harmonics. *Proc. R. Soc. Lond.* A **360**, 471–488.

PEREGRINE, D. H. 1981 Refraction of finite-amplitude water waves: deep-water waves approaching circular caustics. *J. Fluid Mech.* **109**, 63–74.

PEREGRINE, D. H. 1983 Wave jumps and caustics in the refraction of finite-amplitude water waves. *J. Fluid Mech.* (to appear).

PEREGRINE, D. H. & SMITH, R. 1979 Nonlinear effects upon waves near caustics. *Phil. Trans. R. Soc. Lond.* A **292**, 341–370.

PEREGRINE, D. H. & THOMAS, G. P. 1979 Finite-amplitude deep-water waves on currents. *Phil. Trans. R. Soc. Lond.* A **292**, 371–390.

RYRIE, S. & PEREGRINE, D. H. 1982 Refraction of finite-amplitude water waves obliquely incident to a uniform beach. *J. Fluid Mech.* **115**, 91–104.

SAFFMAN, P. G. & YUEN, H. C. 1980 A new type of three-dimensional deep-water wave of permanent form. *J. Fluid Mech.* **101**, 797–808.

STIASSNIE, M. & PEREGRINE, D. H. 1980 Shoaling of finite-amplitude surface waves on water of slowly-varying depth. *J. Fluid Mech.* **97**, 783–805.

WHITHAM, G. B. 1965 Non-linear dispersive waves. *Proc. R. Soc. Lond.* A **283**, 238–261.

WHITHAM, G. B. 1974 *Linear and Nonlinear Waves.* Wiley.